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SMALL $\mathfrak{u}_\kappa$ AND LARGE 2^κ FOR SUPERCOMPACT κ

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Abstract. Garti and Shelah [2] state that one can force $\mathfrak{u}_\kappa$ to be κ^+ for supercompact κ with 2^κ arbitrarily large, using the technique of Džamonja and Shelah [1]. Here we spell out how this can be done.

§1. Introduction. For any regular cardinal λ, we let

$$\mathfrak{u}_\lambda = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a filter base for a uniform ultrafilter on } \lambda\}$$

(recall that an ultrafilter is *uniform* if every set in it has the same cardinality). A simple diagonalisation argument shows that $\mathfrak{u}_\lambda$ must be at least λ^+. In [2], Garti and Shelah state that for κ a supercompact cardinal, one can force $\mathfrak{u}_\kappa = \kappa^+$ with 2^κ arbitrarily large. They provide a short proof sketch, appealing to the arguments of [1]. We give here a detailed proof, based on the pair of talks the author gave in the Kobe University set theory seminar on the topic, closely following [1]. It should be noted that we have not discussed this with Shelah or Garti, so what is presented might not exactly match their original intention, but it seems (to the author) to be the most natural way to proceed.

We base our notation on that of Džamonja and Shelah [1], but do change much of it. A particularly important change to note is that we use $p \leq q$ to mean that p is a stronger condition than q, in contrast with the usage in [1].

The intention is that this note should be readable with no prior knowledge of [1] or [2].

§2. The partial order. Let κ be a supercompact cardinal, and take $\Upsilon \geq 2^\kappa$ such that $\Upsilon^\kappa = \Upsilon$. We will exhibit a forcing that makes $\mathfrak{u}_\kappa = \kappa^+$ and $2^\kappa = \Upsilon$. To this end, we shall actually describe a forcing iteration of length Υ^+, which can be truncated at an appropriate point to obtain the desired forcing (Garti and Shelah [2] mention an

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iteration of length κ^+; with Džamonja and Shelah's loose use of the word "iteration" in [1], this matches the cofinality κ^+ iteration we present).

We use the natural generalisation of Mathias forcing at measurable κ rather than ω (or alternatively put, the natural generalisation of Prikry forcing to obtain κ sequences rather than those of length ω; we shall only ever be concerned with ultrafilters). That is, for $\mathcal{D}$ an ultrafilter on κ, conditions in $\mathbb{M}^\kappa_\mathcal{D}$ are pairs (s, X) such that $s \in [\kappa]^{<\kappa}$ and $X \in \mathcal{D}$, and $(t, Y) \leq (s, X)$ if and only if t end-extends s and $(t \smallsetminus s) \cup Y \subseteq X$.

We define an iteration $\langle P_i, \dot{Q}_i : i < \Upsilon^+\rangle$ as follows. Let G_i be P_i-generic; we describe Q_i in $V[G_i]$. Let NUF denote the set of normal ultrafilters on κ (in the measurable sense — we only need supercompactness of κ to give us a Laver diamond — see below). The partial order Q_i is then the sum over $\mathcal{D} \in$ NUF (interpreted in $V[G_i]$) of the partial orders $\mathbb{M}^\kappa_\mathcal{D}$. That is, we take a maximum element $\mathbb{1}_{Q_i}$ (Džamonja and Shelah use $\emptyset$), and set

$$Q_i = \{\mathbb{1}_{Q_i}\} \cup \bigcup \{\{\mathcal{D}\} \times \mathbb{M}^\kappa_\mathcal{D} : \mathcal{D} \in \mathrm{NUF}\},$$

with $p \leq q$ if and only if either

1. $q = \mathbb{1}_{Q_i}$, or
2. there are $\mathcal{D} \in$ NUF and $p_1 \leq q_1 \in \mathbb{M}^\kappa_\mathcal{D}$ such that $p = (\mathcal{D}, p_1)$ and $q = (\mathcal{D}, q_1)$.

We shall write $\mathbb{1}_\mathcal{D}$ for $(\mathcal{D}, (\emptyset, \kappa))$, the maximum element of the $\mathcal{D}$ part of Q_i.

Now to the support of elements of P_i, $i \leq \Upsilon^+$. We define the *essential support of* p, $\mathrm{SDom}(p)$, by

$$\mathrm{SDom}(p) = \big\{j \in \mathrm{dom}(p) : \\ \neg \left(p \restriction j \Vdash_{P_j} p(j) \in \{\mathbb{1}_{Q_j}\} \cup \{\mathbb{1}_\mathcal{D} : \mathcal{D} \in \mathrm{NUF}\}\right) \big\}.$$

Thus, $\mathrm{SDom}(p)$ is the set of coordinates at which p does something more than just choosing the ultrafilter for forcing at that stage. We require that conditions in P_{Υ^+} have support bounded below Υ^+ and essential support of cardinality strictly less than κ. We freely identify P_{Υ^+} with $\bigcup_{i<\Upsilon^+} P_i$.

We call a condition $p \in P_i$ *purely full in* P_i or *purely full in its domain* if for all $j < i$ we have

$$p \restriction j \Vdash_{P_j} p(j) \in \{\mathbb{1}_\mathcal{D} : \mathcal{D} \in \mathrm{NUF}\}.$$

For $p \in P_i$ we write $P_i \downarrow p$ for $\{q \in P_i : q \leq p\}$; we will particularly be interested in the case when p is purely full in P_i.

LEMMA 1 (Claim 1.13 of [1]). *P_{Υ^+} is κ-directed-closed.*

PROOF. Each $\mathbb{M}^\kappa_{\mathcal{D}}$ is κ-directed-closed, so this is standard. ⊣

LEMMA 2 (Claim 1.16 of [1]). *Let τ be a P_{Υ^+}-name and suppose that p purely full in P_i forces that τ names a set in the ground model V. Then there is a $q \leq p$ purely full in its domain and a $(P_{\mathrm{dom}(q)} \downarrow q)$-name σ such that $q \Vdash \tau = \sigma$.*

PROOF. This is essentially just the κ^+ chain condition. Suppose the Lemma fails for p purely full in its domain and τ a P_{Υ^+}-name. By recursion on $\zeta < \Upsilon^+$ we define $i_\zeta \in \Upsilon^+$, $\sigma_\zeta \in V^{P_{i_\zeta}}$, p_ζ purely full in P_{i_ζ}, $r_\zeta \in P_{i_{\zeta+1}} \downarrow p_{\zeta+1}$, and $x_\zeta \in V$, such that:

1. $\langle i_\zeta \mid \zeta < \Upsilon^+\rangle$ is strictly increasing continuous,
2. $\langle p_\zeta \mid \zeta < \Upsilon^+\rangle$ is decreasing, with $\mathrm{dom}(p_\zeta) = i_\zeta$ and $p_0 = p$,
3. $r_\zeta \Vdash \tau = \check{x}_\zeta$, and $r_\zeta \perp r_\xi$ for all $\xi < \zeta$,
4. $\sigma_\zeta = \{\langle \check{w}, r_\xi\rangle : \xi < \zeta \text{ and } w \in x_\xi\}$.

Suppose we have p_ξ, i_ξ and σ_ξ for all $\xi \leq \zeta$, and r_ξ and x_ξ for $\xi < \zeta$, satisfying 1–4. From our assumption that the Lemma fails we have that $p_\zeta \nVdash \tau = \sigma_\zeta$. Using (3) and (4), this means that $\{r_\xi : \xi < \zeta\}$ is not predense below p_ζ. So there is some $r_\zeta \leq p_\zeta$ in P_{Υ^+} incompatible with each r_ξ, $\xi < \zeta$. By extending if necessary, we may arrange that there is some specific $x_\zeta \in V$ such that $r_\zeta \Vdash \tau = \check{x}_\zeta$, and that $\mathrm{dom}(r_\zeta) = \sup(\mathrm{dom}(r_\zeta))$. We may then define $p_{\zeta+1}$, $i_{\zeta+1}$ and $\sigma_{\zeta+1}$ from r_ζ. Since continuity determines the values of i_ζ, p_ζ and σ_ζ for ζ a limit ordinal, this completes the recursive definition.

But now $\{r_\zeta : \zeta < \Upsilon^+\}$ is an antichain lying in $\bigcup_{\zeta<\Upsilon^+} P_{i_\zeta} \downarrow p_\zeta$. This suborder of P_{Υ^+} is essentially the same as the $<\kappa$-support iteration with α-th iterand $\mathbb{M}^\kappa_{p^*(\alpha)}$ for every $\alpha < \Upsilon^+$, where $p^* = \bigcup_{\zeta<\Upsilon^+} p_\zeta$. Džamonja and Shelah formalise this, and observe that this latter iteration is κ^+-cc, but perhaps it is easiest here to simply observe that the same proof (a Δ-system argument on essential supports) shows that $\bigcup_{\zeta<\Upsilon^+} P_{i_\zeta} \downarrow p_\zeta$ is also κ^+-cc. In any case, we have a contradiction. ⊣

LEMMA 3. *For $\Upsilon \leq j < \Upsilon^+$ and p purely full in P_j, $P_j \downarrow p$ has a dense suborder of cardinality Υ.*

PROOF. This is again a use of the κ^+-cc, along with the fact that $\Upsilon^\kappa = \Upsilon$. We argue by induction. For any $i < j$, a condition in Q_i

below $p(i)$ consists of a sequence from κ of length less than κ and a subset of κ, all of which may be determined by κ many antichains from P_i. At limit stages, the result follows from the fact that we are using $< \kappa$ (essential) support. $\dashv$

§3. Isolating an appropriate suborder. Having defined $P_{\Upsilon+}$ and observed some basic properties, we now move to the key task of isolating a suborder that will be what we actually force with to get $u_\kappa < 2^\kappa$. This suborder will be of the form $P_\alpha \downarrow p$ for some condition p purely full in P_α; the task thus boils down to constructing an appropriate p.

Since $P_{\Upsilon+}$ is κ-directed-closed (Lemma 1), it is natural to first apply a Laver preparation [3] to ensure that κ remains supercompact after our forcing. For our argument, we will actually use it to obtain much more. So let $h : \kappa \to V_\kappa$ be a Laver diamond, and let $\langle S_\alpha, \dot{R}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ denote the Laver preparation defined using h [3]. That is, S_κ is a reverse Easton iteration, and the sequence $\langle \dot{R}_\beta : \beta < \kappa\rangle$ and an auxiliary sequence of ordinals $\langle \lambda_\beta : \beta < \kappa\rangle$ are defined recursively according the the Laver diamond: if $\beta > \lambda_\gamma$ for all $\gamma < \beta$, and $h(\beta)$ is an ordered pair with first term a P_β-name for a β-directed-closed partial order and second term an ordinal, then we set $(\dot{R}_\beta, \lambda_\beta) = h(\beta)$; otherwise, we take R_β to be the trivial forcing and λ_β to be 0.

Take $\lambda \geq |S_\kappa * \dot{P}_{\Upsilon+}|$ (this is probably overkill, but it makes no difference), and let $j : V \to M$ with ${}^\lambda M \subseteq M$ be a λ-supercompactness embedding with critical point κ sent to $j(\kappa) > \lambda$, such that $j(h)(\kappa) = (P_{\Upsilon+}, \lambda)$. In particular, applying j to the Laver preparation S_κ we get $j(S_\kappa) = S^M_{j(\kappa)} = S_\kappa * \dot{P}_{\Upsilon+} * \dot{S^*}^M$ for the appropriate tail iteration S^* (in M). Let us denote $j(P_{\Upsilon+})$ by $P'_{j(\Upsilon+)}$. Thus, applying j to $S_\kappa * \dot{P}_{\Upsilon+}$ yields

$$j(S_\kappa * \dot{P}_{\Upsilon+}) = S_\kappa * \dot{P}_{\Upsilon+} * \dot{S}^* * \dot{P}'_{j(\Upsilon+)}{}^M.$$

In the definition of the Laver preparation, if we have a non-trivial iterand $\dot{R}_\alpha$ coming from $h(\alpha) = (\dot{R}_\alpha, \lambda_\alpha)$, then the subsequent iterands used are trivial until at least stage $\lambda_\alpha + 1$, and thereafter must be at least $|\lambda_\alpha|$-directed-closed. Since direct limits are only taken at inaccessible stages, it follows that the tail of the iteration from stage $\alpha+1$ onward is at least $|\lambda_\alpha|$-directed closed. In particular, we have in M that $\dot{S}^*$ is forced to be at least λ-directed-closed. By elementarity, we also have that $P'_{j(\Upsilon+)}$ is $j(\kappa) > \lambda$-directed-closed.

Main Claim (1.18 of [1]). *In V^{S_κ}, there exist sequences*

$$\bar\alpha = \langle \alpha_i : i < \Upsilon^+\rangle,$$
$$\bar p^* = \langle p_i^* : i < \Upsilon^+\rangle, \text{ and}$$
$$\bar q^* = \langle q_i^* = ({}^1q_i, {}^2q_i) : i < \Upsilon^+\rangle,$$

such that the following hold.

a. $\bar\alpha$ is a strictly increasing continuous sequence of ordinals less than Υ^+.
b. Each p_i^* is purely full in P_{α_i+1}.
c. $\bar p^*$ is a decreasing sequence of conditions in P_{Υ^+}.
d. $\bar q^* \in M^{S_\kappa}$, and in M^{S_κ} we have for each $i < \Upsilon^+$ that
$$(p_i^*, {}^1q_i) \in P_{\Upsilon^+} * \dot S^*$$
and
$$(p_i^*, {}^1q_i, {}^2q_i) \in P_{\Upsilon^+} * \dot S^* * \dot P'_{j(\alpha_i+1)}.$$
e. In M^{S_κ}, $\langle (p_i^*, {}^1q_i, {}^2q_i) : i < \Upsilon^+\rangle$ is a decreasing sequence of conditions in $P_{\Upsilon^+} * \dot S^* * \dot P'_{\sup_{i<\Upsilon^+}(j(\alpha_i+1))}$.
f. In M^{S_κ}, $(p_{i+1}^*, {}^1q_{i+1})$ forces that ${}^2q_{i+1}$ is a common extension of
$$\{j(r) : r \in G_{P_{\alpha_i+1}}\}$$
g. If $\dot B$ is an S_κ-name for a P_{α_i+1}-name for a subset of κ then there is an $S_\kappa * \dot P_{\Upsilon^+}$-name $\tau_{\dot B}$ for an element of $\{0,1\}$ such that:
(1) in V, $(\mathbb{1}_{S_\kappa}, \dot p_{i+1}^*)$ forces $\tau_{\dot B}$ to be a $P_{\alpha_{i+1}+1} \downarrow p_{i+1}^*$-name, and
(2) $M \models [(\mathbb{1}_{S_\kappa}, \dot p_{i+1}^*, q_{i+1}^*) \Vdash \check\kappa \in j(\dot B) \leftrightarrow \tau_{\dot B} = \check 1]$.
i. If $\mathrm{cf}(i) > \kappa$, then in $V^{S_\kappa * \dot P_{\alpha_i}}$ we have that
$$p_i^*(\alpha_i) = \left\{ \dot B[G_{P_{\alpha_i}}] : \begin{array}{l} \dot B \text{ is a } P_{\alpha_i} \downarrow (p_i^* \restriction \alpha_i)\text{-name for a subset} \\ \text{of } \kappa \text{ and } \tau_{\dot B}[G_{P_{\alpha_i}}] = 1 \end{array} \right\}.$$
In particular, this is a normal ultrafilter on κ.

(We have omitted (h) from our labelling so that it corresponds to that in [1].)

The crucial idea here is buried in item (g.2) We have an elementary embedding with critical point κ, and we want a nice normal ultrafilter on κ, so as ever we define it by saying that $B \subseteq \kappa$ is in the ultrafilter if and only if κ is in $j(B)$. In this context "κ is in $j(B)$" must be reinterpreted as "κ is forced to be in $j(\dot B)$", but these statements can be decided by boundedly much of the forcing P_{Υ^+}, as demonstrated by appeal to the technical device of the names $\tau_{\dot B}$. In typical fashion, a long enough iteration with bookkeeping to ensure that every name for a subset of κ is dealt with "catches up with itself". At such closure stages we have that the resulting ultrafilter is defined purely in terms of the construction that came before, and in particular does not require a generic for the forcing $S^* * \dot{P}'_{j(\Upsilon^+)}$ for its definition. Moreover, these particular ultrafilters cohere with (indeed, extend) one another, allowing us to describe an ultrafilter in the final extension in terms of those that came before, and to arrange that $\mathfrak{u}_\kappa = \kappa^+ < 2^\kappa$ (see Theorem 1 below).

Proof of Main Claim. Whilst the statement of the Main Claim might at first seem onerous, the sequences $\bar{\alpha}$, $\bar{p}^*$ and $\bar{q}^*$ can actually be obtained by a relatively natural recursive construction, making used of the λ-directed-closure of S^* and $P'_{j(\Upsilon^+)}$ noted above. Indeed (a)–(e) merely set out the form of the sequences, and whilst there is something to check, (i) is actually giving part of the definition for us. Thus, the key to the recursive construction is ensuring that (f) and (g) hold. We could begin with $\alpha_0 = 0$, p_0^* an arbitrary purely full element of P_1 (that is, $p_0^* = \mathbb{1}_{\mathcal{D}}$ for some arbitrary $\mathcal{D}$ in $\mathrm{NUF}^{V^{S_\kappa}}$), and $q_0^* = (\mathbb{1}_{S^*}, \dot{\mathbb{1}}_{P'_{j(\Upsilon^+)}})$. But it will be notationally convenient if every α_i has cardinality Υ, so let us take $\alpha_0 = \Upsilon$, p_0^* an arbitrary purely full element of $P_{\Upsilon+1}$, and $q_0^* = (\mathbb{1}_{S^*}, \dot{\mathbb{1}}_{P'_{j(\Upsilon^+)}})$.

Choice of α_{i+1}, p_{i+1}^* and q_{i+1}^*, given α_i and p_i^* in V^{S_κ}. First, towards the satisfaction of (f), note that since M is closed under taking λ-tuples, we have

$$\dot{X}_i = \{\langle j(\check{r}), r\rangle : r \in P_{\alpha_i+1} \downarrow p_i^*\}^{V^{S_\kappa}} \in M^{S_\kappa}.$$

Of course, for generics containing p_i^*, this $\dot{X}_i$ names $j``G_{P_{\alpha_i+1}}$, and

$$(p_i^*, \mathbb{1}_{S^*}) \Vdash_{P_{\alpha_i+1}} \dot{X}_i \subseteq \dot{P}'_{j(\alpha_i)+1} \downarrow j(\check{p}_i^*) \wedge \\ \dot{X}_i \text{ is directed} \wedge |\dot{X}_i| \leq \check{\Upsilon}.$$

Thus, the λ-directed-closure of $P'_{j(\alpha_i)+1}$ allows us to find a master condition extending every condition in X_i, giving us the means to satisfy (f). We postpone the use of this, as we will need to interleave it with our construction towards the satisfaction of (g).

In V^{S_κ}, we have that $P_{\alpha_i+1} \downarrow p_i^*$ is a κ^+-cc partial order of size Υ (see Lemma 3, so there are $(\Upsilon^\kappa)^\kappa = \Upsilon$ nice $P_{\alpha_i+1} \downarrow p_i^*$ names for subsets of κ. Enumerate them in order type Υ as $\langle \dot{B}_\zeta^{i+1} : \zeta < \Upsilon\rangle$. To choose p_{i+1}^* and q_{i+1}^*, we perform a further recursive construction, defining

$$\begin{aligned}
\langle \alpha^{i+1}_\zeta : \zeta < \Upsilon\rangle &\text{ increasing continuous,}\\
\langle p^{i+1}_\zeta : \zeta < \Upsilon\rangle &\text{ decreasing continuous}\\
&\text{ with each } p^{i+1}_\zeta \text{ purely full in } P_{\alpha^{i+1}_\zeta},\\
\langle q^{i+1}_\zeta = ({}^1q^{i+1}_\zeta, {}^2q^{i+1}_\zeta) : \zeta < \Upsilon\rangle &\text{ (forced to be) decreasing, and}\\
\langle \tau_{\dot{B}^{i+1}_\zeta} : \zeta < \Upsilon\rangle &\text{ a sequence of } S_\kappa * \dot{P}_{\Upsilon^+}\text{-names}\\
&\text{ for elements of } \{0,1\}.
\end{aligned}$$

Notice in particular that, whilst $\mathrm{dom}(p^*_i) = \alpha_i+1$, $\mathrm{dom}(p^{i+1}_\zeta) = \alpha^{i+1}_\zeta$. Naturally enough, we start this recursion with $p^{i+1}_0 = p^*_i$ and $q^{i+1}_0 = q^*_i$.

Given p^{i+1}_ζ and q^{i+1}_ζ, we want to extend to $p^{i+1}_{\zeta+1}$ and $q^{i+1}_{\zeta+1}$ in a way that "deals with" $\dot{B}^{i+1}_\zeta$. We ask whether there exists a q that forces κ into $j(\dot{B}^{i+1}_\zeta)$ and which acts as a master condition for what has come before (perhaps confusingly, the *negation* of this query is referred to as "the ζ question" in [1]). Let us make this precise.

We work in $M[G_{S_\kappa * \dot{P}_{\alpha^{i+1}_\zeta}}]$, for some generic $G_{S_\kappa * \dot{P}_{\alpha^{i+1}_\zeta}} \ni (\mathbb{1}_S, \dot{p}^{i+1}_\zeta)$. The values of q and $\tau'_{\dot{B}^{i+1}_\zeta}$ that we describe there can then be combined below corresponding conditions in P_{Υ^+} to get single P_{Υ^+}-names in the usual way.

We let

$$X^{i+1}_\zeta = \{j(r) : r \in G_{S_\kappa * \dot{P}_{\alpha^{i+1}_\zeta}}\};$$

as for X_i, this will be in $M[G_{S_\kappa * \dot{P}_{\alpha^{i+1}_\zeta}}]$.

In $M[G_{S_\kappa * \dot{P}_{\alpha^{i+1}_\zeta}}]$, we ask whether there is a condition $q = ({}^1q, {}^2q)$ in $S^* * \dot{P}'_{j(\Upsilon^+)}$ such that

(α) $q \leq q^{i+1}_\zeta$ (and hence by induction $q \leq q^{i+1}_\xi$ for all $\xi \leq \zeta$), and
(β)

$$\begin{aligned}
{}^1q \Vdash_{S^*} \forall r \in \dot{X}^{i+1}_\zeta ({}^2q \leq r) \wedge&\\
{}^2q \in \dot{P}'_{j(\alpha^{i+1}_\zeta)} \downarrow j(p^{i+1}_\zeta) \wedge&\\
{}^2q \Vdash \check{\kappa} \in j(\dot{B}^{i+1}_\zeta).&
\end{aligned}$$

Of course, the first conjunct in (β) is towards making (f) hold, and the second conjunct is also to this end, ensuring that ${}^2q$ does not interfere with parts of j"G that arise later. The final conjunct is, obviously, towards the satisfaction of (g).

Case 1. Suppose there is no q that satisfies both (α) and (β). Then we define $\tau'_{\dot{B}^{i+1}_\zeta}$ to be 0 (in $M[G_{S_\kappa * \dot{P}_{\alpha^{i+1}_\zeta}}]$; in M this of course contributes to the definition of a $S_\kappa * \dot{P}_{\alpha^{i+1}_\zeta}$-name).

We claim that it is possible to find q satisfying all of (α) and (β) except for the final conjunct of (β), and take $q^{i+1}_{\zeta+1} = ({}^1q^{i+1}_{\zeta+1}, {}^2q^{i+1}_{\zeta+1})$ to be such a condition. That is, we take $q^{i+1}_{\zeta+1} \leq q^{i+1}_\zeta$ such that

$$(\beta') \qquad {}^1q^{i+1}_{\zeta+1} \Vdash \forall r \in \dot{X}^{i+1}_\zeta ({}^2q^{i+1}_{\zeta+1} \leq r) \wedge {}^2q^{i+1}_{\zeta+1} \in \dot{P}'_{j(\alpha^{i+1}_\zeta)} \downarrow j(p^{i+1}_\zeta).$$

An appropriate condition can be found since, as in the case of X_i above, $(p^{i+1}_0, \dot{1}_{S^*})$ forces $\dot{X}^{i+1}_\zeta$ to be small and directed, and by induction, the constraint on the support of ${}^2q$ in (β) and (β') ensures that all previous ${}^2q^{i+1}_\xi$ are compatible with everything in $\dot{X}^{i+1}_\zeta$. (Džamonja and Shelah mention that $\dot{X}_i$ is in fact forced to be κ-directed — indeed, it is a simple exercise to show that this is true for the generic of any $< \kappa$-strategically closed forcing. But of course the κ in "κ-directed-closed" refers to an upper bound on the size of the set, not the level of directedness, and so directedness suffices for our purposes.)

Case 2. If there is a q satisfying (α) and (β), then we take $q^{i+1}_{\zeta+1}$ to be such a q, and take $\tau'_{\dot{B}^{i+1}_\zeta} = 1$.

Stepping back to M^{S_κ} now, we may reconstruct $P_{\Upsilon^+} \downarrow p^{i+1}_\zeta$-names $q^{i+1}_{\zeta+1}$ and $\tau'_{\dot{B}^{i+1}_\zeta}$. Since $\tau'_{\dot{B}^{i+1}_\zeta}$ is a P_{Υ^+}-name for an element of the ground model, by Lemma 2 there is a purely full in its domain $p^{i+1}_{\zeta+1} \leq p^{i+1}_\zeta$ with domain some $\alpha^{i+1}_{\zeta+1}$ such that $p^{i+1}_{\zeta+1}$ forces $\tau'_{\dot{B}^{i+1}_\zeta}$ to be equivalent to a $P_{\alpha^{i+1}_{\zeta+1}} \downarrow p^{i+1}_{\zeta+1}$-name; let us take $\tau_{\dot{B}^{i+1}_\zeta}$ to be such a name. This concludes the description of the choice of $p^{i+1}_{\zeta+1}$, $\alpha^{i+1}_{\zeta+1}$, $q^{i+1}_{\zeta+1}$, and $\tau_{\dot{B}^{i+1}_\zeta}$.

For limit ordinals $\zeta < \Upsilon$, we take $\alpha^{i+1}_\zeta = \sup_{\xi<\zeta}(\alpha^{i+1}_\xi)$ and $p^{i+1}_\zeta = \bigcup_{\xi<\zeta} p^{i+1}_\xi$. Once again using the fact that $S^* * \dot{P}'_{j(\Upsilon^+)}$ is Υ^+-directed-closed, we may take q^{i+1}_ζ to be a lower bound for $\{q^{i+1}_\xi : \xi < \zeta\}$. This concludes the recursion defining the sequences $\langle \alpha^{i+1}_\zeta \rangle$, $\langle p^{i+1}_\zeta \rangle$,

$\langle q^{i+1}_\zeta\rangle$, and $\langle \tau_{\dot B^{i+1}_\zeta}\rangle$. We now define $\alpha_{i+1} = \sup_{\zeta<\Upsilon}(\alpha^{i+1}_\zeta)$, take p^*_{i+1} to be any purely full condition in $P_{\alpha_{i+1}+1}$ extending $\bigcup_{\zeta<\Upsilon} p^{i+1}_\zeta$ (so it is only $p^*_{i+1}(\alpha_{i+1})$ that is arbitrary), and take $q^*_{i+1} \in S^* * \dot P'_{j(\alpha_{i+1})}$ such that

$$(\mathbb{1}_{S_\kappa}, p^*_{i+1}) \Vdash \forall\zeta < \Upsilon(q^*_{i+1} \le q^{i+1}_\zeta).$$

By construction, the requirements of the Main Claim (most notably items (f) and (g)) are satisfied by these choices.

It remains to consider the choice of α_i, p^*_i, and q^*_i for $i < \Upsilon^+$ a limit ordinal. Clearly we must take $\alpha_i = \sup_{j<i}\alpha_j$. Likewise we must take p^*_i purely full extending $\bigcup_{j<i} p{*}_j$, only leaving open the question of $p^*_i(\alpha_i)$: if $\mathrm{cf}(i) \ge \kappa$, we take $p^*_i(\alpha_i)$ as given by item (i) of the Main Claim, and otherwise we take p^*_i arbitrary. We similarly take q^*_i to be (forced to be) an arbitrary common extension in $S^* * \dot P'_{j(\alpha_i)}$ of q^*_j for every $j < i$, as well as of every element of X_i; yet again this is possible by the level of (directed) closure of $S^* * \dot P'_{j(\alpha_i)}$. The key items (f) and (g) of the Main Claim only deal with successor stages, so all that remains to check is that item (i) indeed yields a normal ultrafilter on κ when $\mathrm{cf}(i) > \kappa$. As Džamonja and Shelah note, this is a fairly straightforward incorporation of master conditions into the usual normal-ultrafilter-from-an-embedding argument. We shall nevertheless spell it out further.

First note that the definition of $p^*_i(\alpha_i)$ makes sense: if $\dot B$ is a $P_{\alpha_i} \downarrow (p^*_i \restriction \alpha_i)$-name for a subset of κ, it is (equivalent to) a P_{α_j}-name for some $j < i$. This follows from the κ^+-cc of $P_{\alpha_i} \downarrow (p^*_i \restriction \alpha_i)$ (noted in the proof of Lemma 2) and the fact that $\mathrm{cf}(i) > \kappa$.

Suppose $G_{S_\kappa * \dot P_{\alpha_i}}$ is an $S_\kappa * \dot P_{\alpha_i} \downarrow (\mathbb{1}_{S_\kappa}, p^*_i \restriction \alpha_i)$ generic, and that in $V[G_{S*\dot P_{\alpha_i}}]$, $A \in p^*_i(\alpha_i)$ and $B \supseteq A$; we wish to show that B is also in $p^*_i(\alpha_i)$. Choose names $\dot A$ and $\dot B$ for A and B respectively, and let $j < i$ be such that both $\dot A$ and $\dot B$ are $P_{\alpha_j+1} \downarrow p^*_j$-names. Suppose $p \in G_{S*\dot P_{\alpha_i}}$ forces $A \in p^*_i(\alpha_i)$ and $B \supseteq A$, that is,

$$(\dagger) \qquad p \Vdash \tau_{\dot A} = 1 \wedge \dot B \supseteq \dot A.$$

By extending if necessary we may assume that $p \le (\mathbb{1}_{S_\kappa}, p^*_{j+1})$. Thus by item (g.2) of the Main Claim, in M we have that

$$(\ddagger) \qquad (p, q^*_{j+1}) \Vdash (\check\kappa \in j(\dot A) \leftrightarrow \tau_{\dot A} = 1) \wedge (\check\kappa \in j(\dot B) \leftrightarrow \tau_{\dot B} = 1).$$

It should be clear how we proceed from here, but note in particular the following point: to deduce from $(p, q^*_{j+1}) \Vdash \check\kappa \in j(\dot A)$ and

$p \Vdash \dot{B} \supseteq \dot{A}$ that $(p, q^*_{j+1}) \Vdash \check{\kappa} \in j(\dot{B})$, we need that j lifts to an elementary embedding between the relevant forcing extensions. This is precisely why we needed to extend to master conditions at every step of the iteration. To be explicit, $p \Vdash \dot{B} \supseteq \dot{A}$ implies by elementarity only that $j(p) \Vdash j(\dot{B}) \supseteq j(\dot{A})$. However, item (f) of the Main Claim ensures that $(p, q^*_{j+1}) \leq j(p)$, so combining this with (†) and (‡) we can indeed conclude that

$$(p, q^*_{j+1}) \Vdash \tau_{\dot{B}} = 1.$$

But now $\tau_{\dot{B}}$ is a P_{Υ^+}-name, so it must be that $p \Vdash \tau_{\dot{B}} = 1$, and so $B \in p^*_i(\alpha_i)$.

The rest of the process of checking that $p^*_i(\alpha_i)$ is a normal ultrafilter on κ is very similar, using the fact that we have taken master conditions to get

$$\begin{aligned}
&\text{from } p \Vdash \dot{B} = \check{\kappa} \smallsetminus \dot{A} && \text{to } (p, q^*_{j+1}) \Vdash \check{\kappa} \notin j(\dot{A}) \leftrightarrow \check{\kappa} \in j(\dot{B}),\\
&\text{from } p \Vdash \dot{B} = \bigcap_{\gamma<\delta} \dot{A}_\gamma && \text{to } (p, q^*_{j+1}) \Vdash \forall \gamma < \delta(\check{\kappa} \in j(\dot{A}_\gamma)) \to \check{\kappa} \in j(\dot{B}),\\
\&\ &\text{from } p \Vdash \dot{B} = \mathop{\triangle}_{\gamma<\kappa} \dot{A}_\gamma && \text{to } (p, q^*_{j+1}) \Vdash \forall \gamma < \kappa(\check{\kappa} \in j(\dot{A}_\gamma)) \to \check{\kappa} \in j(\dot{B}).
\end{aligned}$$

So we indeed have a normal ultrafilter, and hence a valid definition for $p^*_i(\alpha_i)$ for i a limit ordinal of cofinality greater than κ. This completes the proof of the Main Claim. ⊣

With the Main Claim in hand we can finally prove the following.

THEOREM 1. *Let κ be a supercompact cardinal, and let $\Upsilon \geq 2^\kappa$ be a cardinal satisfying $\Upsilon^\kappa = \Upsilon$. Then there is a forcing extension in which κ remains supercompact, $\mathfrak{u}_\kappa = \kappa^+$, and $2^\kappa = \Upsilon$.*

PROOF. With $\langle \alpha_i : i < \Upsilon^+ \rangle$ as in the Main Claim, we take the forcing $S_\kappa * \dot{P}_{\alpha_i} \downarrow (p^*_i \restriction \alpha_i)$ for $i = \kappa^+ \cdot \kappa^+$ (the ordinal square of κ^+). Let G be $S_\kappa * \dot{P}_{\alpha_i} \downarrow (p^*_i \restriction \alpha_i)$-generic over V. Since we begin with the Laver preparation, κ certainly remains supercompact in $V[G]$, and since $|\alpha_{\kappa^+ \cdot \kappa^+}| = \Upsilon$, $2^\kappa = \Upsilon$ in $V[G]$.

To show that $\mathfrak{u}_\kappa = \kappa^+$ in the generic extension, we consider the normal ultrafilter given by item (i) of the Main Claim, which would be $p^*_{\kappa^+ \cdot \kappa^+}(\alpha_{\kappa^+ \cdot \kappa^+})$ if we continued the iteration. That is, in $V[G]$ we consider

$$\mathcal{D} = \{B \subset \kappa : \tau_{\dot{B}}[G] = 1\}$$

(of course, whether B is in $\mathcal{D}$ is independent of the choice of $\dot{B}$ by the construction of the names $\tau_{\dot{B}}$). Since conditions in $P_{\alpha_i} \downarrow (p_i^* \restriction \alpha_i)$ have essential support bounded below $\alpha_{\kappa^+ \cdot \kappa^+}$, each subset B of κ in the extension is named by some stage of the iteration prior to $\alpha_{\kappa^+ \cdot \kappa^+}$. In particular, either B or $\kappa \setminus B$ will appear in the ultrafilter $p^*_{\kappa^+ \cdot \delta}(\alpha_{\kappa^+ \cdot \delta})[G]$ for $\delta < \kappa^+$ sufficiently large, and this ultrafilter is determined by item (i) of the Main Claim. Hence, we have that $B \in p^*_{\kappa^+ \cdot \delta}(\alpha_{\kappa^+ \cdot \delta})[G]$ if and only if $\tau_{\dot{B}}[G] = 1$, if and only if $B \in \mathcal{D}$. We thus have that

$$\mathcal{D} = \bigcup_{\delta < \kappa^+} p^*_{\kappa^+ \cdot \delta}(\alpha_{\kappa^+ \cdot \delta})[G].$$

Now at stage $\alpha_{\kappa^+ \cdot \delta}$ of the iteration, we are forcing with $\mathbb{M}^\kappa_{p^*_{\kappa^+ \cdot \delta}}$, and so the generic subset of κ at this stage, $X_{\alpha_{\kappa^+ \cdot \delta}}$, is almost below every element of $p^*_{\kappa^+ \cdot \delta}(\alpha_{\kappa^+ \cdot \delta})[G]$. Hence, $\mathcal{D}$ is generated by the set

$$\{Y \subseteq \kappa : \exists \delta < \kappa^+ (|Y \triangle X_{\alpha_{\kappa^+ \cdot \delta}}| < \kappa)\},$$

which has cardinality κ^+. ⊣

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